Time-frequency duality

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We have seen in the lecture about Fourier transform that the spectrum of rectangle signal $x(t) = R_{[-a,a]}(t)$ is the sinc function $X(\omega) = \frac{2\sin(a\omega)}{\omega} = 2a\,\mathrm{sinc}(a\omega)$. The support of this spectrum is not bounded. More generally, if y is a time-limited signal, i.e. its time support is a bounded interval $[a, b]$, then it can be seen as the product $y(t) = z(t)R_{[a,b]}(t) = z(t)x(t)$ where z may not be a time-limited signal. Applying the multiplication property of Fourier transform, we can write $Y = Z * X$, thus even if spectrum Z has bounded support, signal y will not be frequency-limited. Therefore, a time-limited signal cannot be frequency-limited. Conversely, using the inverse Fourier transform of a rectangle spectrum, we show that a frequency-limited signal cannot be time-limited. The extreme case is the Dirac delta function δ which is exactly located at time $t = 0$ but whose spectrum $\omega \mapsto 1$ is spread over all the frequencies. The goal of this lecture is to prove a lower bound on the product of time and frequency spreadings called the **Heisenberg-Gabor** inequality.
We first recall the following notions:

- The **energy** of a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ is

$$E(x) = \|x\|^2 = \langle x, x\rangle = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

- **Plancherel's identity**: Let a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ whose Fourier transform is denoted $X = \mathcal{F}(x)$. Then

$$E(x) = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

Remarks:

- Plancherel's identity exhibits a relation between the energy in the time domain and the energy in the frequency domain, represented by the two integrals.
- Consider a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ with non-zero energy $E(x) \neq 0$. By Plancherel's identity,

$$1 = \int_{-\infty}^{+\infty} \frac{|x(t)|^2}{E(x)} dt = \int_{-\infty}^{+\infty} \frac{|X(\omega)|^2}{2\pi E(x)} d\omega$$

  Therefore, we can interpret functions $t \mapsto \frac{|x(t)|^2}{E(x)}$ and $\omega \mapsto \frac{|X(\omega)|^2}{2\pi E(x)}$ as probability density functions. The objective of this lecture is to establish a relation between the spreadings of the energy in time and frequency, i.e. the standard deviations of these density functions.

Definition 0.1 (Average position, spreading)
Let a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ with non-zero energy and Fourier transform X. We define:

- the average position in time:

$$m_t(x) = \int_{-\infty}^{+\infty} t\frac{|x(t)|^2}{E(x)} dt$$

- the spreading in time:

$$\sigma_t(x) = \left(\int_{-\infty}^{+\infty} (t - m_t(x))^2 \frac{|x(t)|^2}{E(x)} dt \right)^{\frac{1}{2}}$$

- the average position in frequency:

$$m_\omega(x) = \int_{-\infty}^{+\infty} \omega \frac{|X(\omega)|^2}{2\pi E(x)} d\omega$$

- the spreading in frequency:

$$\sigma_\omega(x) = \left(\int_{-\infty}^{+\infty} (\omega - m_\omega(x))^2 \frac{|X(\omega)|^2}{2\pi E(x)} d\omega \right)^{\frac{1}{2}}$$

Lemma 0.1
Let a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ with non-zero energy. We define the following signal:

$$\forall t \in \mathbb{R} \qquad y(t) = x\left(t + m_t(x)\right) e^{-im_\omega(x)t}$$

This signal y satisfies the following properties:

- it has the same energy as x: $E(y) = E(x)$;
- it has a zero average position in time and frequency: $m_t(y) = 0$ and $m_\omega(y) = 0$;
- it has the same spreadings as x in time and frequency: $\sigma_t(y) = \sigma_t(x)$ and $\sigma_\omega(y) = \sigma_\omega(x)$.

Proof : By definition of the energy, using the identity $|e^{-im_\omega(x)t}|^2 = 1$ and the change of variable $t \mapsto t + m_t(x)$, we get:

$$E(y) = \int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |x\left(t + m_t(x)\right) e^{-im_\omega(x)t}|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt = E(x)$$

We have:

$$m_t(y) = \int_{-\infty}^{+\infty} t \frac{|y(t)|^2}{E(y)} dt = \int_{-\infty}^{+\infty} t \frac{|x\left(t + m_t(x)\right)|^2}{E(x)} dt$$

By the change of variable $t \mapsto t + m_t(x)$, we get:

$$m_t(y) = \int_{-\infty}^{+\infty} (t - m_t(x)) \frac{|x(t)|^2}{E(x)} dt = \int_{-\infty}^{+\infty} t \frac{|x(t)|^2}{E(x)} dt - m_t(x) \int_{-\infty}^{+\infty} \frac{|x(t)|^2}{E(x)} dt = 0$$

To compute the average position of y in frequency, we determine its Fourier transform Y. We set $z : t \mapsto x(t + m_t(x))$ so that $y : t \mapsto z(t)e^{-im_\omega(x)t}$. Then we have $Z(\omega) = X(\omega)e^{im_t(x)\omega}$, and

$$Y(\omega) = Z(\omega + m_\omega(x)) = X(\omega + m_\omega(x))e^{im_t(x)(\omega + m_\omega(x))}$$

We deduce:

$$m_\omega(y) = \int_{-\infty}^{+\infty} \omega \frac{|Y(\omega)|^2}{2\pi E(y)} d\omega = \int_{-\infty}^{+\infty} \omega \frac{|X(\omega + m_\omega(x))|^2}{2\pi E(x)} d\omega = 0$$

where we used the same reasoning as in the computation of $m_t(y)$.
Finally, by the change of variable $t \mapsto t + m_t(x)$ we have:

$$\sigma_t(y) = \left(\int_{-\infty}^{+\infty} t^2 \frac{|y(t)|^2}{E(y)} dt\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} t^2 \frac{|x(t+m_t(x))|^2}{E(x)} dt\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} (t-m_t(x))^2 \frac{|x(t)|^2}{E(x)} dt\right)^{\frac{1}{2}} = \sigma_t(x)$$

Using the Fourier transform of y, we have:

$$\sigma_\omega(y) = \left(\int_{-\infty}^{+\infty} \omega^2 \frac{|Y(\omega)|^2}{2\pi E(y)} d\omega\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} \omega^2 \frac{|X(\omega+m_\omega(x))|^2}{2\pi E(x)} d\omega\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} (\omega-m_\omega(x))^2 \frac{|X(\omega)|^2}{2\pi E(x)} d\omega\right)^{\frac{1}{2}} = \sigma_\omega(x)$$

which concludes the proof of the lemma. ■

Theorem 0.2 (Heisenberg-Gabor inequality)
For any signal $x \in L^2(\mathbb{R}, \mathbb{K})$ with non-zero energy,

$$\sigma_t(x)\sigma_\omega(x) \geq \frac{1}{2}$$

Proof : Using the previous lemma, we have:

$$\sigma_t(x)\sigma_\omega(x) = \sigma_t(y)\sigma_\omega(y) = \frac{1}{E(y)} \left(\int_{-\infty}^{+\infty} t^2 |y(t)|^2 dt\right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 |Y(\omega)|^2 d\omega\right)^{\frac{1}{2}}$$

Since the Fourier transform of derivative y' is $\mathcal{F}(y') : \omega \mapsto i\omega Y(\omega)$, applying Plancherel's identity, we get:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{F}(y')(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} |y'(t)|^2 dt$$

yielding

$$\sigma_t(x)\sigma_\omega(x) = \frac{1}{E(y)} \left(\int_{-\infty}^{+\infty} |ty(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} |y'(t)|^2 dt\right)^{\frac{1}{2}}$$

Using Cauchy-Schwarz inequality, we obtain:

$$\sigma_t(x)\sigma_\omega(x) \geq \frac{1}{E(y)} \left|\int_{-\infty}^{+\infty} ty(t)\overline{y'(t)} dt\right|$$

Since the modulus of a complex number is greater or equal to its real part, we have

$$\left|\int_{-\infty}^{+\infty} ty(t)\overline{y'(t)} dt\right| \geq \left|\mathrm{Re}\left(\int_{-\infty}^{+\infty} ty(t)\overline{y'(t)} dt\right)\right| = \left|\int_{-\infty}^{+\infty} t\mathrm{Re}\left(y(t)\overline{y'(t)}\right) dt\right|$$

Since $\mathrm{Re}\left(y(t)\overline{y'(t)}\right)$ is the derivative of $\frac{|y(t)|^2}{2}$, integration by parts gives:

$$\left|\int_{-\infty}^{+\infty} t\mathrm{Re}\left(y(t)\overline{y'(t)}\right) dt\right| = \frac{1}{2} \int_{-\infty}^{+\infty} |y(t)|^2 dt = \frac{E(y)}{2}$$

yielding the expected result. ■

Remarks:

- If we want to use frequency f instead of $\omega = 2\pi f$, we have:

$$m_f(x) = \int_{-\infty}^{+\infty} f \frac{|X(\omega)|^2}{E(x)} df = \frac{m_\omega(x)}{2\pi} \qquad \text{and} \qquad \sigma_f(x) = \left(\int_{-\infty}^{+\infty} (f - m_f(x))^2 \frac{|X(\omega)|^2}{2E(x)} df \right)^{\frac{1}{2}} = \frac{\sigma_\omega(x)}{2\pi}$$

  Then the Heisenberg-Gabor inequality becomes:

$$\sigma_t(x)\sigma_f(x) \geq \frac{1}{4\pi}$$

- We look for the signals reaching the lower bound of the Heisenberg-Gabor inequality. According to the proof, such signals must be real-valued and they must satisfy the Cauchy-Schwarz equality, i.e. there exists a constant $\alpha \in \mathbb{R}$ such that $y'(t) = \alpha t y(t)$. The solutions of this differential equation are the functions of the form $y(t) = Ke^{\frac{\alpha t^2}{2}}$. When $\alpha < 0$, we recognize the gaussian functions. The existence of these functions shows that the lower bound is optimal.